Indefinite locally conformal Kähler manifolds

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ABSTRACT. We study the basic properties of an indefinite locally conformal Kähler (l.c.K.) manifold. Any indefinite l.c.K. manifold M with a parallel Lee form ω is shown to possess two canonical foliations $\mathcal{F}$ and $\mathcal{F}_c$, the first of which is given by the Pfaff equation $\omega = 0$ and the second is spanned by the Lee and the anti-Lee vectors of M. We build an indefinite l.c.K. metric on the noncompact complex manifold $\Omega_+ = (\Lambda_+ \setminus \Lambda_0)/G_\lambda$ (similar to the Boothby metric on a complex Hopf manifold) and prove a CR extension result for CR functions on the leafs of $\mathcal{F}$ when $M = \Omega_+$ (where $\Lambda_+ \setminus \Lambda_0 \subset \mathbb{C}^n_s$ is $-|z_1|^2 - \cdots - |z_s|^2 + |z_{s+1}|^2 + \cdots + |z_n|^2 > 0$). We study the geometry of the second fundamental form of the leaves of $\mathcal{F}$ and $\mathcal{F}_c$. In the degenerate cases (corresponding to a lightlike Lee vector) we use the technique of screen distributions and (lightlike) transversal bundles developed by A. Bejancu et al., [10].

1. The first canonical foliation

Let M be a complex n-dimensional indefinite Hermitian manifold of index $0 < \nu < 2n$, with the complex structure J and the semi-Riemannian metric g. As well known ν must be even, $\nu = 2s$. M is an *indefinite Kähler manifold* if $\nabla J = 0$, where ∇ is the Levi-Civita connection of (M, g), cf. M. Baros & A. Romero, [1]. An indefinite Hermitian manifold M is an *indefinite locally conformal Kähler* (l.c.K.) *manifold* if for any point $x \in M$ there is an open neighborhood U of x in M and a C^∞ function $f : U \to \mathbb{R}$ such that $(U, e^{-f} g)$ is an indefinite Kähler manifold.

Note that any two conformally related indefinite Kähler metrics are actually homothetic. Indeed, let $(U, z^1, \cdots, z^n)$ be a local system of complex coordinates on M and set $g_{j\bar{k}} = g(Z_j, \overline{Z}_k)$, where Z_j is short for $\partial/\partial z^j$ (and overbars denote complex conjugates). If $\hat{g} = e^f g$ then the Levi-Civita connections ∇ and $\hat{\nabla}$ (of (M, g) and $(M, \hat{g})$, respectively)

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are related by

$$\hat{\nabla}_{Z_j}\overline{Z}_k = \nabla_{Z_j}\overline{Z}_k - \frac{1}{2}\{(Z_j f)\overline{Z}_k + (\overline{Z}_k f)Z_j - g_{j\overline{k}}\nabla f\}$$

where ∇f is the gradient of f with respect to g, i.e. $g(\nabla f, X) = X(f)$, for any $X \in T(M)$. Let $T(M)\otimes\mathbb{C}$ be the complexified tangent bundle. Let $Z^{1,0}$ denote the $(1,0)$-component of $Z \in T(M)\otimes\mathbb{C}$ with respect to the direct sum decomposition $T(M)\otimes\mathbb{C} = T^{1,0}(M)\oplus T^{0,1}(M)$, where $T^{1,0}(M)$ is the holomorphic tangent bundle over M and $T^{0,1}(M) = \overline{T^{1,0}(M)}$. When g and $\hat{g}$ are indefinite Kähler metrics, both ∇ and $\hat{\nabla}$ descend to connections in $T^{0,1}(M)$ hence

$$(\overline{Z}_k f)Z_j - g_{j\overline{k}}(\nabla f)^{1,0} = 0$$

or $\delta_j^s \overline{Z}_k f - g_{j\overline{k}} g^{s\overline{r}}\overline{Z}_r f = 0$. Contraction of s and j leads to $(n-1)\overline{Z}_k f = 0$, i.e. f is a real valued holomorphic function, hence a constant.

Let (M, J, g) be an indefinite l.c.K. manifold. Let then $\{U_i\}_{i\in I}$ be an open cover of M and $\{f_i\}_{i\in I}$ a family of C^∞ functions $f_i : U_i \to \mathbb{R}$ such that $g_i := e^{-f_i}g$ is an indefinite Kähler metric on U_i. Then $g_j = e^{f_i - f_j}g_i$ on $U_i \cap U_j$, i.e. g_i and g_j are conformally related indefinite Kähler metrics on $U_i \cap U_j$, hence $f_i - f_j = c_{ij}$, for some $c_{ij} \in \mathbb{R}$. In particular $df_i = df_j$, i.e. the local 1-forms df_i, $i \in I$, glue up to a globally defined closed 1-form ω on M. By analogy with the positive definite case (cf. e.g. S. Dragomir & L. Ornea, [7]) we shall refer to ω as the *Lee form* of M. An indefinite l.c.K. metric g is called *globally conformal Kähler* (g.c.K.) if the Lee form ω is exact (cf. e.g. [18] for the Riemannian case). The tangent vector field B on M defined by $g(X, B) = \omega(X)$, for any $X \in T(M)$, is the *Lee field*. Let us set $c = g(B, B) \in C^\infty(M)$ and $\mathrm{Sing}(B) = \{x \in M : B_x = 0\}$. Note that (opposite to the positive definite case) it may be $c = 0$ and $\mathrm{Sing}(B) = \emptyset$ (when B is lightlike).

Let ∇^i be the Levi-Civita connection of (U_i, g_i), $i \in I$. Then

$$\nabla^i_X Y = \nabla_X Y - \frac{1}{2}\{X(f_i)Y + Y(f_i)X - g(X,Y)\nabla f_i\}$$

for any $X, Y \in T(U_i)$, hence the local connections ∇^i, $i \in I$, glue up to a globally defined linear connection D on M given by

$$D_X Y = \nabla_X Y - \frac{1}{2}\{\omega(X)Y + \omega(Y)X - g(X,Y)B\},$$

the *Weyl connection* of M. Clearly $DJ = 0$.

Let us analyze indefinite l.c.K. manifolds with $\nabla\omega = 0$ (the indefinite counterpart of generalized Hopf manifolds, cf. I. Vaisman, [18]). Such manifolds carry a natural foliation $\mathcal{F}$ defined by the Pfaff equation

$\omega = 0$. Also $c \in \mathbb{R}$, so that B is spacelike (respectively timelike, or lightlike) when $c > 0$ (respectively $c < 0$, or $c = 0$). We shall prove

Theorem 1. *Let M be a complex n-dimensional indefinite l.c.K. manifold of index $2s$, $0 < s < n$, with a parallel Lee form and with* $\mathrm{Sing}(B) = \emptyset$. *Then either* i) $c \neq 0$ *and then each leaf L of $\mathcal{F}$ is a totally geodesic semi-Riemannian hypersurface of (M, g) of index*

$$\mathrm{ind}(L) = \begin{cases} 2s & c > 0 \\ 2s - 1 & c < 0. \end{cases} \tag{1}$$

or ii) $c = 0$ *and then each leaf of $\mathcal{F}$ is a totally geodesic lightlike hypersurface of (M, g).*

Proof. Assume first that $c \neq 0$. Let us show that

$$T(M) = T(\mathcal{F}) \oplus \mathbb{R}B. \tag{2}$$

To this end, let $X \in T(M)$. Then $X - \frac{1}{c}\omega(X)B \in T(\mathcal{F})$. Moreover, if $X \in T(\mathcal{F}) \cap \mathbb{R}B$ then $X = \lambda B$, for some $\lambda \in C^\infty(M)$, and $0 = \omega(X) = \lambda c$ yields $\lambda = 0$, i.e. $X = 0$. Therefore (2) holds. Since $\mathbb{R}B$ is nondegenerate it follows that $T(\mathcal{F}) = (\mathbb{R}B)^\perp$ is nondegenerate, as well, hence each leaf of $\mathcal{F}$ is a semi-Riemannian hypersurface of (M, g) and (1) holds. Let L be a leaf of $\mathcal{F}$. Let ∇^L be the induced connection and h^L the second fundamental form of $i : L \hookrightarrow M$. If $X, Y \in T(L)$ then, by $\nabla\omega = 0$ and the Gauss formula $\nabla_X Y = \nabla^L_X Y + h^L(X, Y)$

$$0 = X(\omega(Y)) = \omega(\nabla_X Y) = \omega(h^L(X, Y))$$

hence $h^L = 0$.

Let us assume now that $c = 0$, so that $B \in T(\mathcal{F})$. Let us set as customary (cf. e.g. [10], p. 140)

$$\mathrm{Rad}\; T(\mathcal{F})_x = T(\mathcal{F})_x \cap T(\mathcal{F})_x^\perp, \quad x \in M.$$

Clearly $B \in \mathrm{Rad}\; T(\mathcal{F})$. Note that $\dim_{\mathbb{R}} T(\mathcal{F})_x = 2n - 1$ hence (cf. e.g. Proposition 2.2 in [10], p. 6) $\dim_{\mathbb{R}} T(\mathcal{F})_x^\perp = 1$, for any $x \in M$. Therefore, if $\mathrm{Sing}(B) = \emptyset$ then $T(\mathcal{F})^\perp = \mathbb{R}B$ and (by Proposition 1.1 in [10], p. 78) each leaf of $\mathcal{F}$ is a lightlike hypersurface of (M, g).

If $Y \in T(\mathcal{F})^\perp$ then Y is orthogonal on the Lee field, hence $Y \in T(\mathcal{F})$. It follows that $T(\mathcal{F})^\perp \subset T(\mathcal{F})$, i.e. $\mathrm{Rad}\; T(\mathcal{F}) = T(\mathcal{F})^\perp$. Let $S(T\mathcal{F})$ be a distribution on M such that

$$T(\mathcal{F}) = S(T\mathcal{F}) \oplus_{\mathrm{orth}} T(\mathcal{F})^\perp. \tag{3}$$

If V is a semi-Euclidean space and $W_a \subset V$, $a \in \{1, 2\}$, are two subspaces then we write $V = W_1 \oplus_{\mathrm{orth}} W_2$ whenever $V = W_1 \oplus W_2$ and the subspaces W_a are mutually orthogonal. According to the terminology

in [10], p. 78, the portion of $S(T\mathcal{F})$ over a leaf L of $\mathcal{F}$ is a *screen distribution* on L. The choice of $S(T\mathcal{F})$ is not unique, yet (by Proposition 2.1 in [10], p. 5) $S(T\mathcal{F})$ is nondegenerate, hence

$$T(M) = S(T\mathcal{F}) \oplus_{\rm orth} S(T\mathcal{F})^{\perp}. \tag{4}$$

Note that $S(T\mathcal{F})^{\perp}$ has rank two and $T(\mathcal{F})^{\perp} \subset S(T\mathcal{F})^{\perp}$. The following result is an adaptation (to the foliation $\mathcal{F}$ of M, rather than a single lightlike hypersurface) of Theorem 1.1 in [10], p. 79.

Lemma 1. *Let $\pi : E \to M$ be a subbundle of $S(T\mathcal{F})^{\perp} \to M$ such that $S(T\mathcal{F})^{\perp} = T(\mathcal{F})^{\perp} \oplus E$. Let $V \in \Gamma^{\infty}(U, E)$ be a locally defined nowhere zero section, defined on the open subset $U \subseteq M$. Then* i) $\omega(V) \neq 0$ *everywhere on U. Let us consider $N_V \in \Gamma^{\infty}(U, S(T\mathcal{F})^{\perp})$ given by*

$$N_V = \frac{1}{\omega(V)} \left\{ V - \frac{g(V,V)}{2\omega(V)}\, B \right\}. \tag{5}$$

If $V' \in \Gamma^{\infty}(U', E)$ is another nowhere zero section, defined on the open subset $U' \subseteq M$ such that $U \cap U' \neq \emptyset$, then ii) $N_V = N_{V'}$ *on $U \cap U'$. Moreover, let $x \in M$ and $U \subseteq M$ an open neighborhood of x such that $E|_U = \pi^{-1}(U)$ is trivial. Let V us set*

$$\mathrm{tr}(T\mathcal{F})_x = \mathbb{R} N_V(x)\,. \tag{6}$$

Then iii) $\mathrm{tr}(T\mathcal{F})_x$ *is well defined and gives a lightlike subbundle* $\mathrm{tr}(T\mathcal{F}) \to M$ *of $S(T\mathcal{F})^{\perp} \to M$ such that*

$$S(T\mathcal{F})^{\perp} = T(\mathcal{F})^{\perp} \oplus \mathrm{tr}(T\mathcal{F}). \tag{7}$$

Finally, iv) *the definition of* $\mathrm{tr}(T\mathcal{F})$ *doesn't depend upon the choice of complement E to $T(\mathcal{F})^{\perp}$ in $S(T\mathcal{F})^{\perp}$.*

Proof. The proof of (i) is by contradiction. If $\omega(V)_{x_0} = 0$ for some $x_0 \in U$ then $V_{x_0} \in T(\mathcal{F})_{x_0}$ and then (by (3))

$$V_{x_0} \in S(T\mathcal{F})_{x_0} \cap S(T\mathcal{F})^{\perp}_{x_0} = (0),$$

a contradiction. To prove (ii) let V' be a nowhere zero section in E on U' (with $U \cap U' \neq \emptyset$). Then $V' = \alpha V$, for some C^{∞} function $\alpha : U \cap U' \to \mathbb{R} \setminus \{0\}$, and an inspection of (5) leads to $N_V = N_{V'}$ on $U \cap U'$. This also shows that the definition of $\mathrm{tr}(T\mathcal{F})_x$ doesn't depend upon the particular local trivialization chart of E at x. To check the remaining statement in (iii) note first that

$$g(N_V, N_V) = 0, \quad \omega(N_V) = 1. \tag{8}$$

The first relation in (8) shows that $\mathrm{tr}(T\mathcal{F})$ is lightlike, while the second relation yields $T(\mathcal{F})^{\perp} \cap \mathrm{tr}(T\mathcal{F}) = (0)$. Yet both bundles have rank one, hence (7) holds. Finally, if $F \to M$ is another complement to $T(\mathcal{F})^{\perp}$

in $S(T\mathcal{F})^\perp$ then an *a priori* new lightlike bundle, similar to $\mathrm{tr}(T\mathcal{F})$, may be build in terms of a local section $W \in \Gamma^\infty(U, F)$. Yet (by (7)) $W = \alpha N_V + \beta B$, for some $\alpha, \beta \in C^\infty(U)$, hence (by (8)) $N_W = N_V$. □

According to the terminology in [10], p. 79, the portion of $\mathrm{tr}(T\mathcal{F})$ over a leaf L of $\mathcal{F}$ is the *lightlike transversal vector bundle* of L with respect to the screen distribution $S(T\mathcal{F})|_L$. By (3)-(4) and (7) we obtain the decomposition

$$T(M) = S(T\mathcal{F}) \oplus_{\mathrm{orth}} \left[T(\mathcal{F})^\perp \oplus \mathrm{tr}(T\mathcal{F})\right] = T(\mathcal{F}) \oplus \mathrm{tr}(T\mathcal{F}). \tag{9}$$

Let $\tan : T(M) \to T(\mathcal{F})$ and $\mathrm{tra} : T(M) \to \mathrm{tr}(T\mathcal{F})$ be the projections associated with (9). Next, we set

$$\nabla^{\mathcal{F}}_X Y = \tan\left(\nabla_X Y\right), \quad h(X,Y) = \mathrm{tra}\left(\nabla_X Y\right),$$

$$A_V X = -\tan\left(\nabla_X V\right), \quad \nabla^{\mathrm{tr}}_X V = \mathrm{tra}\left(\nabla_X V\right),$$

for any $X, Y \in T(\mathcal{F})$ and any $V \in \mathrm{tr}(T\mathcal{F})$. Then $\nabla^{\mathcal{F}}$ is a connection in $T(\mathcal{F}) \to M$, h is a symmetric $\mathrm{tr}(T\mathcal{F})$-valued bilinear form on $T(\mathcal{F})$, A_V is an endomorphism of $T(\mathcal{F})$, and ∇^{tr} is a connection in $\mathrm{tr}(T\mathcal{F}) \to M$. Also one has

$$\nabla_X Y = \nabla^{\mathcal{F}}_X Y + h(X,Y), \quad \nabla_X V = -A_V X + \nabla^{\mathrm{tr}}_X V,$$

the *Gauss* and *Weingarten formulae* of $\mathcal{F}$ in (M, g). Clearly, the point-wise restrictions of $\nabla^{\mathcal{F}}$, ∇^{tr}, h and A_V to a leaf L of $\mathcal{F}$ are respectively the induced connections, the second fundamental form and the shape operator of L in (M, g), cf. [10], p. 83. A leaf L is *totally geodesic* if each geodesic of $\nabla^{\mathcal{F}}$ lying on L is also a geodesic of the semi-Riemannian manifold (M, g).

Let us prove the last statement in Theorem 1. As $\nabla\omega = 0$ it follows that $\omega(h(X,Y)) = 0$. Yet locally (with the notations in the proof of Lemma 1) $h(X,Y) = C(X,Y)N_V$, for some $C(X,Y) \in C^\infty(U)$, hence (by (8)) $h = 0$ and then by a result in [2] (cf. also Theorem 2.2 in [10], p. 88) each leaf of $\mathcal{F}$ is totally geodesic in (M, g). □

We end this section by the following remark. By Theorem 2.2 in [10], p. 88, if $c = 0$ then $\nabla^{\mathcal{F}}$ is the Levi-Civita connection of the tangential metric induced by g on $T(\mathcal{F})$ and the distribution $T(\mathcal{F})^\perp$ is Killing.

2. Indefinite Hopf manifolds

Let $\mathbb{C}^n_s$ denote $\mathbb{C}^n$ together with the real part of the Hermitian form

$$b_{s,n}(z, w) = -\sum_{j=1}^{s} z_j \overline{w}_j + \sum_{j=s+1}^{n} z_j \overline{w}_j\,, \quad z, w \in \mathbb{C}^n.$$

Let $\Lambda = \{z \in \mathbb{C}^n \setminus \{0\} : -\sum_{j=1}^{s} |z_j|^2 + \sum_{j=s+1}^{n} |z_j|^2 = 0\}$ be the null cone in $\mathbb{C}^n_s$ and $\Lambda_0 = \Lambda \cup \{0\}$. Given $\lambda \in \mathbb{C} \setminus \{0\}$

$$F_\lambda(z) = \lambda z, \quad z \in \mathbb{C}^n \setminus \Lambda_0\,,$$

is a holomorphic transformation of $\mathbb{C}^n \setminus \Lambda_0$. Let $G_\lambda = \{F_\lambda^m : m \in \mathbb{Z}\}$ be the discrete group generated by F_λ. Then

Theorem 2. *Let $n > 1$, $0 < s < n$ and $\lambda \in \mathbb{C} \setminus \{0\}$, $|\lambda| \neq 1$. Then G_λ acts freely on $\mathbb{C}^n \setminus \Lambda_0$ as a properly discontinuous group of holomorphic transformations, hence the quotient space $\mathbb{C}H^n_s(\lambda) = (\mathbb{C}^n \setminus \Lambda_0)/G_\lambda$ is a complex manifold and*

$$g_{s,n} = |z|_{s,n}^{-2} \left(-\sum_{j=1}^{s} dz^j \odot d\overline{z}^j + \sum_{j=s+1}^{n} dz^j \odot d\overline{z}^j \right) \tag{10}$$

(where $|z|_{s,n} = |b_{s,n}(z,z)|^{1/2}$) is a globally defined semi-Riemannian metric, making $\mathbb{C}H^n_s(\lambda)$ into an indefinite locally conformal Kähler manifold. Moreover, if $0 < \lambda < 1$ then $\mathbb{C}H^n_s(\lambda) \approx \Sigma^{2n-1} \times S^1$ (a diffeomorphism), where $\Sigma^{2n-1} = \{z \in \mathbb{C}^n : |z|_{s,n} = 1\}$. In particular $\mathbb{C}H^n_s(\lambda)$ is noncompact. If $\Lambda_+ = \{z \in \mathbb{C}^n : b_{s,n}(z,z) \geq 0\}$ and $\Lambda_- = \{z \in \mathbb{C}^n : b_{s,n}(z,z) \leq 0\}$ (so that $\partial\Lambda_\pm = \Lambda_0$) then $\mathbb{C}H^n_s(\lambda)$ consists of the two connected components $(\Lambda_+ \setminus \Lambda_0)/G_\lambda \approx S^{2n-1}_{2s} \times S^1$ and $(\Lambda_- \setminus \Lambda_0)/G_\lambda \approx H^{2n-1}_{2s-1} \times S^1$.

If $\mathbb{R}^N_\nu = (\mathbb{R}^N, h_{\nu,N})$, with $h_{\nu,N}(x,y) = -\sum_{j=1}^{\nu} x_j y_j + \sum_{j=\nu+1}^{N} x_j y_j$, then $S^N_\nu(r) = \{x \in \mathbb{R}^{N+1} : h_{\nu,N+1}(x,x) = r^2\}$ $(r > 0)$ is the pseudosphere in $\mathbb{R}^{N+1}_\nu$, while $H^N_\nu(r) = \{x \in \mathbb{R}^{N+1} : h_{\nu+1,N+1}(x,x) = -r^2\}$ $(r > 0)$ is the pseudohyperbolic space in $\mathbb{R}^{N+1}_{\nu+1}$. When $r = 1$ we write simply S^N_ν and H^N_ν. Also $\odot$ denotes the symmetric tensor product, e.g. $\alpha \odot \beta = \frac{1}{2}(\alpha \otimes \beta + \beta \otimes \alpha)$ for any 1-forms α and β. A construction similar to that in Theorem 2 was performed in [8] (for metrics which are locally conformal to anti-Kählerian metics, cf. Lemma 3, *op. cit.*, p. 119).

Proof of Theorem 2. If $F_\lambda^m(z) = z$ for some $z \in \mathbb{C}^n \setminus \Lambda_0$ then $m = 0$, hence G_λ acts freely on $\mathbb{C}^n \setminus \Lambda_0$. Given $z_0 \in \mathbb{C}^n \setminus \Lambda_0$ let $B_r(z_0)$ be the open Euclidean ball of center z_0 and radius r. Also, we set $\Omega_r(z_0) = B_r(z_0) \setminus \Lambda_0$. As well known (cf. [12], Vol. II, p. 137) G_λ acts on $\mathbb{C}^n \setminus \{0\}$ as a properly discontinuous group of holomorphic transformations, hence there is $r > 0$ such that $F_\lambda^m(B_r(z_0)) \cap B_r(z_0) = \emptyset$, and then $F_\lambda^m(\Omega_r(z_0)) \cap \Omega_r(z_0) = \emptyset$, for any $m \in \mathbb{Z} \setminus \{0\}$. As $\mathbb{C}^n \setminus \Lambda_0$ is an open subset of $\mathbb{C}^n$ it follows (cf. e.g. [5], p. 97) that $\mathbb{C}H^n_s(\lambda) := (\mathbb{C}^n \setminus \Lambda_0)/G_\lambda$ is a complex manifold.

Assume from now on that $0 < \lambda < 1$. Let $\pi : \mathbb{C}^n \setminus \Lambda_0 \to \mathbb{C}H^n_s(\lambda)$ be the projection. To prove the last statement in Theorem 2 we consider

the C^∞ diffeomorphism

$$F : \mathbb{C}H^n_s(\lambda) \to \Sigma^{2n-1} \times S^1,$$

$$F(\pi(z)) = \left(|z|^{-1}_{s,n}\, z\ ,\ \exp\left(\frac{2\pi i \log |z|_{s,n}}{\log \lambda} \right) \right), \tag{11}$$

with the obvious inverse

$$F^{-1}(\zeta, w) = \pi\left(\lambda^{\arg(w)/(2\pi)}\, \zeta\right), \quad \zeta \in \Sigma^{2n-1}, \ \ w \in S^1,$$

where $\arg : \mathbb{C} \to [0, 2\pi)$. Finally, note that $\Sigma^{2n-1} \cap \Lambda_+ = S^{2n-1}_{2s}$ and $\Sigma^{2n-1} \cap \Lambda_- = H^{2n-1}_{2s-1}$. $\square$

Proposition 1. *The Lee form of* $(\mathbb{C}H^n_s(\lambda), g_{s,n})$ *is locally given by*

$$\omega = -d \log |z|^2_{s,n}. \tag{12}$$

In particular $\mathbb{C}H^n_s(\lambda)$ *has a parallel Lee form. Let* $\Omega_\pm = (\Lambda_\pm \backslash \Lambda_0)/G_\lambda$ *be the connected components of* $\mathbb{C}H^n_s(\lambda)$ *and* $a(z) = \mathrm{sign}(b_{s,n}(z,z)) = \pm 1$ *for* $z \in \Lambda_\pm \setminus \Lambda_0$*. Then the Lee field* B *of* $(\mathbb{C}H^n_s(\lambda), g_{s,n})$ *is given by*

$$B = -2a(z)\left(z^j \frac{\partial}{\partial z^j} + \overline{z}^j \frac{\partial}{\partial \overline{z}^j} \right). \tag{13}$$

Finally, if $B_\pm = B|_{\Omega_\pm}$ *then* B_+ *is spacelike while* B_- *is timelike.*

Proof. An inspection of (10) leads to (12) and hence to

$$\omega = b_{s,n}(z,z)^{-1} \left\{ \sum_{j=1}^{s} (\overline{z}_j dz^j + z_j d\overline{z}^j) - \sum_{j=s+1}^{n} (\overline{z}_j dz^j + z_j d\overline{z}^j) \right\},$$

with the convention $z_j = z^j$. Next

$$g_{j\overline{k}} = \frac{1}{2}\, |z|^{-2}_{s,n}\, \epsilon_j\, \delta_{jk}$$

(where $\epsilon_j = -1$ for $1 \le j \le s$ and $\epsilon_j = 1$ for $s+1 \le j \le n$) yields (13). Since

$$Z_j\left(|z|^{-2}_{s,n}\right) = -a(z)|z|^{-4}_{s,n}\, \epsilon_j\, \overline{z}_j$$

the identity

$$2 g_{AD} \Gamma^A_{BC} = Z_B(g_{CD}) + Z_C(g_{BD}) - Z_D(g_{BC})$$

leads to

$$\Gamma^\ell_{jk} = -\frac{a(z)}{2|z|^2_{s,n}} \left(\epsilon_j \overline{z}_j \delta^\ell_k + \epsilon_k \overline{z}_k \delta^\ell_j \right), \quad \Gamma^{\overline{\ell}}_{jk} = 0,$$

$$\Gamma^\ell_{j\overline{k}} = \frac{a(z)}{2|z|^2_{s,n}} \left(\epsilon_j \delta_{jk} z^\ell - \epsilon_k z_k \delta^\ell_j \right), \ \ \Gamma^{\overline{\ell}}_{j\overline{k}} = \frac{a(z)}{2|z|^2_{s,n}} \left(\epsilon_j \delta_{jk} \overline{z}^\ell - \epsilon_j \overline{z}_j \delta^\ell_k \right),$$

hence a calculation shows that $\nabla_{Z_j} B = 0$. Finally $g_{s,n}(B,B) = 4a(z)$. $\square$

Given two semi-Riemannian manifolds M and N and a C^∞ submersion $\Pi : M \to N$, we say Π is a *semi-Riemannian submersion* if 1) $\Pi^{-1}(y)$ is a semi-Riemannian submanifold of M for each $y \in N$, and 2) $d_x\Pi : \mathcal{H}_x \to T_{\Pi(x)}(N)$ is a linear isometry of semi-Euclidean spaces, where $\mathcal{H}_x = \mathrm{Ker}(d_x\Pi)^\perp$, for any $x \in M$ (cf. [13], p. 212). Also we recall (cf. [1]) the indefinite complex projective space $\mathbb{C}P_s^{n-1}(k)$. Its underlying complex manifold is the open subset of the complex projective space

$$\mathbb{C}P_s^{n-1}(k) = (\Lambda_+ \setminus \Lambda_0)/\mathbb{C}^* \subset \mathbb{C}P^{n-1} \qquad (\mathbb{C}^* = \mathbb{C} \setminus \{0\}).$$

As to the semi-Riemannian metric of $\mathbb{C}P_s^{n-1}(k)$, let

$$\Pi : S_{2s}^{2n-1}(2/\sqrt{k}) \to \mathbb{C}P_s^{n-1}(k), \quad \Pi(z) = z \cdot \mathbb{C}^* \quad (k > 0)$$

be the *indefinite Hopf fibration*. It is a principal S^1-bundle and S^1 acts on $S_{2s}^{2n-1}(2/\sqrt{k})$ as a group of isometries, hence (by slightly adapting the proof of Proposition E.3 in [3], p. 7, to the semi-Riemannian context) there is a unique semi-Riemannian metric of index $2s$ on $\mathbb{C}P_s^{n-1}(k)$ such that Π is a semi-Riemannian submersion and $\mathbb{C}P_s^{n-1}(k)$ is an indefinite complex space form of (constant) holomorphic sectional curvature k. Again by [1], p. 57, $\mathbb{C}P_s^{n-1}(k)$ is homotopy equivalent to $\mathbb{C}P^{n-1-s}$, hence $\mathbb{C}P_s^{n-1}(k)$ is simply connected. We shall prove

Theorem 3. *Let $D = \{2\pi i a + (\log\lambda)b : a, b \in \mathbb{Z}\}$ $(0 < \lambda < 1)$ and consider the torus $T^1_{\mathbb{C}} = \mathbb{C}/D$. Then $T^1_{\mathbb{C}}$ acts freely on $\mathbb{C}H_s^n(\lambda)$ and*

$$p : \Omega_+ \to \mathbb{C}P_s^{n-1}(4), \quad p(\pi(z)) = z \cdot \mathbb{C}^*,$$

is a principal $T^1_{\mathbb{C}}$-bundle and a semi-Riemannian submersion of Ω_+ (carrying the indefinite l.c.K. metric $g_{s,n}$) onto $\mathbb{C}P_s^{n-1}(4)$. Moreover the complex Hopf manifold $\mathbb{C}H^{n-s}(\lambda)$ (respectively $\mathbb{C}H^s(\lambda)$) is a strong deformation retract of Ω_+ (respectively of Ω_-) hence

$$H_k(\Omega_+;\mathbb{Z}) = \begin{cases} \mathbb{Z} \otimes \mathbb{Z}, & k = 2(n-s), \\ \mathbb{Z}, & k \neq 2(n-s), \end{cases}$$

$$H_k(\Omega_-;\mathbb{Z}) = \begin{cases} \mathbb{Z} \otimes \mathbb{Z}, & k = 2s, \\ \mathbb{Z}, & k \neq 2s, \end{cases}$$

and $\Omega_\pm$ are not simply connected

$$\pi_1(\Omega_+) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & s = n-1, \\ \mathbb{Z}, & s \neq n-1, \end{cases} \quad \pi_1(\Omega_-) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}, & s = 1, \\ \mathbb{Z}, & s \neq 1. \end{cases}$$

Each fibre $p^{-1}(z\cdot\mathbb{C}^*)$, $z\in S^{2n-1}_{2s}$, is tangent to the Lee field B of Ω_+ hence $p:\Omega_+\to\mathbb{C}P^{n-1}_s(4)$ is a harmonic map.

Proof. The action of $T^1_{\mathbb{C}}$ on $\mathbb{C}H^n_s(\lambda)$ is given by

$$\pi(z)\cdot(\zeta+D)=\pi\left(e^{\zeta}\, z\right),\quad z\in\mathbb{C}^n\setminus\Lambda_0,\ \ \zeta\in\mathbb{C}.$$

As $b_{s,n}(e^{\zeta}z,e^{\zeta}z)=e^{2\mathrm{Re}(\zeta)}b_{s,n}(z,z)\neq 0$ the action is well defined. To see that the action is free, let us assume that $\pi(e^{\zeta}z_0)=\pi(z_0)$, for some $z_0\in\mathbb{C}^n\setminus\Lambda_0$. Then $e^{\zeta}_0=\lambda^m z_0$, for some $m\in\mathbb{Z}$, hence $\zeta=m\log\lambda+2k\pi i$, for some $k\in\mathbb{Z}$, i.e. $\zeta+D=0$.

To see that $S^1\to\Omega_+\xrightarrow{p}\mathbb{C}^{n-1}_s(4)$ is a principal bundle let us assume that $p(\pi(z))=p(\pi(z'))$, with $z,z'\in\mathbb{C}^n\setminus\Lambda_0$. Then $z'=\alpha z$, for some $\alpha\in\mathbb{C}^*$. We wish to show that there is a unique $\zeta+D\in T^1_{\mathbb{C}}$ such that $\pi(z')=\pi(z)\cdot(\zeta+D)$. Indeed we may consider $\zeta=\log|\alpha|+i\arg(\alpha)$.

Let $F_t:\Lambda_+\setminus\Lambda_0\to\mathbb{C}^n$, $0\leq t\leq 1$, be given by

$$F_t(z)=((1-t)z'\,,\,z''),\quad z=(z',z'')\in\Lambda_+\setminus\Lambda_0,$$

where $z'=(z_1,\cdots,z_s)$ and $z''=(z_{s+1},\cdots,z_n)$. Then

$$b_{s,n}(F_t(z),F_t(z))=-(1-t)^2|z'|^2+|z''|^2\geq b_{s,n}(z,z)>0$$

hence F_t is $(\Lambda_+\setminus\Lambda_0)$-valued. Therefore F_t induces a homotopy

$$H^+_t:\Omega_+\to\Omega_+,\quad H^+_t(\pi(z))=\pi\left(F_t(z)\right),\quad 0\leq t\leq 1.$$

Let us consider (cf. e.g. [12], Vol. II, p. 137) the complex Hopf manifold $\mathbb{C}H^n(\lambda)=(\mathbb{C}^n\setminus\{0\})/G_\lambda$ and denote by $\pi_0:\mathbb{C}^n\setminus\{0\}\to\mathbb{C}H^n(\lambda)$ the projection. Let $\mathbb{C}H^{n-s}(\lambda)$ be thought of as identified to

$$\{\pi_0(z)\in\mathbb{C}H^n(\lambda):z_1=0,\cdots,z_s=0\}.$$

Note that $\mathbb{C}H^{n-s}(\lambda)\subset\Omega_+$. Also

$$H^+_0=1_{\Omega_+}\,,\quad H^+_1\left(\Omega_+\right)\subset\mathbb{C}H^{n-s}(\lambda),$$

$$H^+_t\big|_{\mathbb{C}H^{n-s}(\lambda)}=i,\quad 0\leq t\leq 1,$$

(where $i:\mathbb{C}H^{n-s}(\lambda)\to\Omega_+$ is the inclusion) hence $\mathbb{C}H^{n-s}(\lambda)$ is a strong deformation retract of Ω_+. Also $H^+_1\circ i=1_{\mathbb{C}H^{n-s}(\lambda)}$ and $H^+:1_{\Omega_+}\simeq i\circ H^+_1$ (i.e. the maps $1_{\mathbb{C}H^{n-s}(\lambda)}$ and $i\circ H^+_1$ are homotopic) so that i,H^+_1 are reciprocal homotopy equivalences, i.e.

$$\Omega_+\simeq\mathbb{C}H^{n-s}(\lambda),\tag{14}$$

(a homotopy equivalence). As well known, (14) implies that

$$i_*:H_k(\mathbb{C}H^{n-s}(\lambda);\mathbb{Z})\approx H_k(\Omega_+;\mathbb{Z}),$$

(a group isomorphism). Therefore, to compute $H_k(\Omega_+;\mathbb{Z})$ it suffices to compute the singular homology of the complex Hopf manifold. This is

an easy exercise in algebraic topology (based on the Künneth formula). Indeed

$$H_k(\mathbb{C}H^n(\lambda);\mathbb{Z}) = \sum_{p+q=k} H_p(S^{2n-1};\mathbb{Z}) \otimes H_q(S^1;\mathbb{Z}) =$$

$$= H_{k-1}(S^{2n-1};\mathbb{Z}) \otimes H_1(S^1;\mathbb{Z}) = \begin{cases} \mathbb{Z}\otimes\mathbb{Z}, & k = 2n, \\ \mathbb{Z}, & k \neq 2n, \end{cases}$$

yielding (12). As to the homotopy groups, again by (14)

$$\pi_k(\Omega_+) \approx \pi_k(\mathbb{C}H^{n-s})$$

(a group isomorphism) and if $n > 1$

$$\pi_k(\mathbb{C}H^n(\lambda)) = \pi_k(S^{2n-1}) \oplus \pi_k(S^1) = \begin{cases} \mathbb{Z}, & k \in \{1, 2n-1\}, \\ 0, & k \notin \{1, 2n-1\}, \end{cases}$$

while if $n = 1$

$$\pi_k(\mathbb{C}H^1(\lambda)) = \pi_k(S^1) \oplus \pi_k(S^1) = \begin{cases} \mathbb{Z}\oplus\mathbb{Z}, & k = 1, \\ 0, & k \neq 1. \end{cases}$$

Let us show now that $p : \Omega_+ \to \mathbb{C}P^{n-1}_s(4)$ is a semi-Riemannian submersion. Let $i : S^{2n-1}_{2s} \to \Lambda_+ \setminus \Lambda_0$ be the inclusion and $V_{z_0} = \mathrm{Ker}(d_{z_0}\Pi)$ the vertical space, $z_0 \in S^{2n-1}_{2s}$. As $\Pi : S^{2n-1}_{2s} \to \mathbb{C}P^{n-1}_s(4)$ is a semi-Riemannian submersion V_{z_0} is nondegenerate, hence the perp space H_{z_0} of V_{z_0} is also nondegenerate and

$$T_{z_0}(S^{2n-1}_{2s}) = H_{z_0} \oplus_{\mathrm{orth}} V_{z_0}.$$

Let $V_{0,\pi(z_0)} = \mathrm{Ker}(d_{\pi(z_0)}p)$ and let $N(S^{2n-1}_{2s})_{z_0}$ be the normal space of i at z_0 (as S^{2n-1}_{2s} has index $2s$ it follows that $N(S^{2n-1}_{2s})_{z_0}$ has index zero).

Lemma 2. *For any* $z_0 \in S^{2n-1}_{2s}$

$$V_{0,\pi(z_0)} = (d_{z_0}\pi)\{N(S^{2n-1}_{2s})_{z_0} \oplus (d_{z_0}i)V_{z_0}\}.$$

Proof. Let $N = z^j Z_j + \overline{z}^j \overline{Z}_j$ be the unit normal on S^{2n-1}_{2s} in $\Lambda_+ \setminus \Lambda_0$, with the flat indefinite Kähler metric

$$g_0 = \sum_{j=1}^n \epsilon_j dz^j \odot d\overline{z}^j.$$

Let U be the tangent vector field on S^{2n-1}_{2s} defined by $(di)U = -JN$, where J is the complex structure on $\mathbb{C}^n$. Let

$$\Pi_0 : \Lambda_+ \setminus \Lambda_0 \to \mathbb{C}P^{n-1}_s(4)$$

be the canonical projection (so that $\Pi = \Pi_0|_{S^{2n-1}_{2s}}$). Then $(d\Pi_0)U = 0$ hence $V_{z_0} = \mathbb{R}U_{z_0}$. In particular, by the commutativity of the diagram

$$\begin{array}{ccccc} \mathbb{C}H^n_s(\lambda) & \supset & \Omega_+ & \xrightarrow{p} & \mathbb{C}P^{n-1}_s(4) \\ & & \pi\uparrow & & \uparrow\Pi \\ \mathbb{C}^n_s & \supset & \Lambda_+\setminus\Lambda_0 & \xleftarrow{i} & S^{2n-1}_{2s} \end{array}$$

it follows that

$$(d_{z_0}\pi)(d_{z_0}i)V_{z_0} \subseteq V_{0,\pi(z_0)}. \tag{15}$$

On the other hand Π_0 is a holomorphic map hence

$$(d_{\pi(z_0)}p)(d_{z_0}\pi)N_{z_0} = (d_{z_0}\Pi_0)J_{z_0}(d_{z_0}i)U_{z_0} =$$
$$= J'_{\Pi_0(z_0)}(d_{z_0}\Pi_0)(d_{z_0}i)U_{z_0} = J'_{\Pi(z_0)}(d_{z_0}\Pi)U_{z_0} = 0,$$

where J' denotes the complex structure on $\mathbb{C}P^{n-1}$. We obtain

$$(d_{z_0}\pi)N(S^{2n-1}_{2s})_{z_0} \subseteq V_{0,\pi(z_0)}. \tag{16}$$

At this point Lemma 2 follows from (15)-(16) and an inspection of dimensions. □

As $|z_0|_{s,t} = 1$ the indefinite scalar product $(\ ,\ )_{z_0}$ induced on $T_{z_0}(\Lambda_+\setminus\Lambda_0)$ by g_{0,z_0} coincides with that induced by $|z|^{-2}_{s,t}\sum_j \epsilon_j dz^j \odot d\overline{z}^j$ at z_0.

Lemma 3. *$V_{0,\pi(z)}$ is nondegenerate, for any $z \in \Lambda_+\setminus\Lambda_0$.*

Proof. Let $A = -JB$, where J is the complex structure on Ω_+. Then $\{A_{\pi(z)}, B_{\pi(z)}\}$ span $V_{0,\pi(z)}$ and (by Proposition 1) both A and B are spacelike. □

By Lemma 3 the perp space $H_{0,\pi(z_0)}$ of $V_{0,\pi(z_0)}$ is also nondegenerate and

$$T_{\pi(z_0)}(\Omega_+) = H_{0,\pi(z_0)} \oplus_{\text{orth}} V_{0,\pi(z_0)}.$$

Let $v \in H_{z_0}$. Then $(d_{z_0}i)v$ is perpendicular on $N(S^{2n-1}_{2s})_{z_0} \oplus (d_{z_0}i)V_{z_0}$. On the other hand $d_{z_0}\pi : T_{z_0}(\Lambda_+\setminus\Lambda_0) \to T_{\pi(z_0)}(\Omega_+)$ is a linear isometry hence (by Lemma 2) $(d_{z_0}\pi)(d_{z_0}i)v$ is perpendicular on $V_{0,\pi(z_0)}$, and then it lies on $H_{0,\pi(z_0)}$. Again by inspecting dimensions we obtain

$$H_{0,\pi(z_0)} = (d_{z_0}\pi)(d_{z_0}i)H_{z_0}. \tag{17}$$

Next, by (17) and by $d_{z_0}\Pi : H_{z_0} \approx T_{\Pi(z_0)}(\mathbb{C}P^{n-1}_s(4))$ (a linear isometry), it follows that

$$d_{\pi(z)}p : H_{0,\pi(z)} \approx T_{\Pi(z)}(\mathbb{C}P^{n-1}_s(4)), \tag{18}$$

a linear isometry for any $z \in S^{2n-1}_s$. We wish to show that (18) actually holds for any $z \in \Lambda_+\setminus\Lambda_0$.

Lemma 4. *The torus $T^1_{\mathbb{C}}$ acts on Ω_+ as a group of isometries of the semi-Riemannian manifold $(\Omega_+, g_{s,n})$.*

Proof. If $g = w + D \in T^1_{\mathbb{C}}$ ($w \in \mathbb{C}$) then the right translation $R_g : \Omega_+ \to \Omega_+$ is a holomorphic map locally given by $z \mapsto ze^w$ hence

$$|ze^w|^{-2}_{s,n} \, \left((dR_g)Z_j \, , \, (dR_g)\overline{Z}_k\right)_{ze^w} =$$
$$= e^{2\mathrm{Re}(w)} \, |z|^{-2}_{s,n} \, b_{s,n}(e^w e_j, e^w e_k) = |z|^{-2}_{s,n} \, \epsilon_j \, \delta_{jk}$$

where $\{e_j : 1 \le j \le n\}$ is the canonical basis in $\mathbb{C}^n$. □

As $p \circ R_g = \text{const.}$, each R_g preserves the vertical spaces V_0. Then (by Lemma 4) R_g preserves the horizontal spaces H_0, as well. Therefore, to complete the proof we must show that for any $z \in \Lambda_+ \setminus \Lambda_0$ there is $g \in T^1_{\mathbb{C}}$ and $z_0 \in S^{2n-1}_{2s}$ such that $\pi(z) = \pi(z_0)g$. Indeed we may consider $g = \log|z|_{s,t} + D$ and $z_0 = |z|^{-1}_{s,t} z$.

To prove the last statement in Theorem 2 we establish

Lemma 5. *Let $z_0 \in \Lambda_+ \setminus \Lambda_0$ and let $j : T^1_{\mathbb{C}} \to \Omega_+$ be the immersion given by $j(\zeta + D) = \pi(z_0) \cdot (\zeta + D)$. Then*

$$(dj) \frac{\partial}{\partial u}\bigg|_{\zeta + D} = -\frac{1}{4} \, e^{\zeta} \, B_{\pi(z_0)} \, ,$$

where $\zeta = u + iv$, hence $j(T^1_{\mathbb{C}})$ is tangent to the Lee field of Ω_+.

This follows easily from (13). Then (by Proposition 3) $j : T^1_{\mathbb{C}} \to \Omega_+$ is a minimal isometric immersion. Therefore p is a semi-Riemannian submersion with minimal fibres, hence a harmonic map (in the sense of [11]). Compare to Theorem 3 in [6], p. 375. □

3. An indefinite l.c.K. metric with nonparallel Lee form

Let $\mathbb{C}_+ = \{w \in \mathbb{C} : \mathrm{Im}(w) > 0\}$ be the upper half space and consider the *Tricerri metric* (cf. [7], p. 24)

$$g_{0,1} = \mathrm{Im}(w)^{-2} \, dw \odot d\overline{w} + \mathrm{Im}(w) \, dz \odot d\overline{z}.$$

$g_{0,1}$ is (by a result in [15]) a (positive definite) g.c.K. metric on $\mathbb{C}_+ \times \mathbb{C}$. We build a family of indefinite l.c.K. metrics of index $0 \le s < n$ containing the Tricerri metric as a limiting case (for $s = 0$ and $n = 1$).

Proposition 2. *Let $0 \le s < n$ and $n \ge 1$. Let $g_{s,n}$ be the indefinite Hermitian metric on $\mathbb{C}_+ \times \mathbb{C}^n_s$ given by*

$$g_{s,n} = \mathrm{Im}(w)^{-2} \, dw \odot d\overline{w} + \mathrm{Im}(w) \sum_{j=1}^{n} \epsilon_j dz^j \odot d\overline{z}^j. \tag{19}$$

Then $g_{s,n}$ is an indefinite globally conformal Kähler metric with a non-parallel Lee form and its Lee field is spacelike. Moreover, let $a \in \mathrm{SL}(3, \mathbb{Z})$ be a unimodular matrix with $\mathrm{Spec}(a) = \{\alpha, \beta, \overline{\beta}\}$, where $\alpha > 1$ and $\beta \in \mathbb{C} \setminus \mathbb{R}$. Let $G_{\alpha,\beta}$ be the group of holomorphic transformations

of $\mathbb{C}_+ \times \mathbb{C}^n_s$ *generated by* $F_0(w,z) = (\alpha w, \beta z)$, $w \in \mathbb{C}_+$, $z \in \mathbb{C}^n$. *Then* $g_{s,n}$ *is* $G_{\alpha,\beta}$*-invariant.*

Proof. Note that we may write (19) as $g_{s,n} = \mathrm{Im}(w) g_0$ where

$$g_0 = \mathrm{Im}(w)^{-3}\, dw \odot d\overline{w} + \sum_{j=1}^{n} \epsilon_j dz^j \odot d\overline{z}^j$$

is an indefinite Hermitian metric whose Kähler 2-form Ω_0 is

$$-i\{\mathrm{Im}(w)^{-3}\, dw \wedge d\overline{w} + \sum_{j=1}^{n} \epsilon_j\, dz^j \wedge d\overline{z}^j\}.$$

Hence $d\Omega_0 = 0$, i.e. g_0 is an indefinite Kähler metric. Thus (19) is an indefinite l.c.K. metric whose Lee form

$$\omega = df = \frac{1}{w - \overline{w}}\,(dw - d\overline{w}) \quad (f = \log \mathrm{Im}(w))$$

is exact. Raising indices we obtain the Lee field

$$B = i\,\mathrm{Im}(w)\left(\frac{\partial}{\partial w} - \frac{\partial}{\partial \overline{w}}\right)$$

so that $g_{s,n}(B,B) = 1$, i.e. B is spacelike. Next, the only surviving coefficients of the Levi-Civita connection ∇ of $(\mathbb{C}_+ \times \mathbb{C}^n_s, g_{s,n})$ are

$$\Gamma^j_{k0} = -\Gamma^j_{k\overline{0}} = -\frac{i}{4}\,\mathrm{Im}(w)^{-1}\,\delta^j_k\,,$$

hence $\nabla_{Z_j} B = \frac{1}{2} Z_j \neq 0$. Finally, the $G_{\alpha,\beta}$-invariance of $g_{s,n}$ follows from $F_0^* dw = \alpha\ dw$, $F_0^* dz^j = \beta\ dz^j$, $1 \leq j \leq n$, and from $\alpha\beta\overline{\beta} = \det(a) = 1$. $\square$

4. The second canonical foliation

Let (M, J, g) be an indefinite l.c.K. manifold, of index $\nu = 2s$. Let $A = -JB$ be the *anti-Lee field*. We also set $\theta(X) = g(X, A)$ (the *anti-Lee form*), so that $\theta = \omega \circ J$. Also, let $\Omega(X,Y) = g(X, JY)$ be the Kähler 2-form. Since $DJ = 0$ it follows that

$$(\nabla_X J)Y = \frac{1}{2}\{\theta(Y)X - \omega(Y)JX - g(X,Y)A - \Omega(X,Y)B\} \tag{20}$$

for any $X, Y \in T(M)$. As an immediate application of (20) we have

Proposition 3. *Let* (M,J,g) *be an indefinite l.c.K. manifold and* $i : N \hookrightarrow M$ *a complex submanifold of* M *such that* i^*g *is a semi-Riemannian metric. Let* h *be the second fundamental form of* i*. Then*

$$h(JX, JY) = -h(X,Y) - g(X,Y)B^{\perp}\,, \tag{21}$$

for any $X, Y \in T(N)$, where $B^\perp$ is the normal component of the Lee field of M. Then the mean curvature vector of i is given by $H = -\frac{1}{2}B^\perp$. In particular i is minimal if and only if N is tangent to the Lee field.

This extends a result of I. Vaisman (cf. [18]) to the case of semi-Riemannian complex submanifolds of an indefinite l.c.K. manifold. *Proof of Proposition* 3. As N is a complex manifold $T(N)$ admits a local orthonormal frame of the form $\{E_\alpha, JE_\alpha : 1 \leq \alpha \leq m\}$, i.e. $g(E_\alpha, E_\beta) = \epsilon_\alpha \delta_{\alpha\beta}$, and then (by (21)) the mean curvature vector H of i is given by

$$H = \frac{1}{2m} \sum_\alpha \epsilon_\alpha \{h(E_\alpha, E_\alpha) + h(JE_\alpha, JE_\alpha)\} = -\frac{1}{2}\, B^\perp.$$

It remains that we prove (21). Let tan and nor be the projections associated with the decomposition $T(M) = T(N) \oplus T(N)^\perp$ and let us set

$$t\xi = \tan(\xi), \quad f\xi = \mathrm{nor}(\xi), \quad \xi \in T(N)^\perp.$$

Then $J\xi = t\xi + f\xi$ and by applying J once more we get $f^2 = -I$. Let $A^\perp = \mathrm{nor}(A)$ and $B^\perp = \mathrm{nor}(B)$. Then (by (20) and the Gauss formula)

$$h(X, JY) = f\ h(X, Y) - \frac{1}{2}\{g(X, Y)A^\perp + \Omega(X, Y)B^\perp\},$$

for any $X, Y \in T(N)$. Finally, using $A^\perp = -f\ B^\perp$ and $f^2 = -I$ we obtain (21). $\square$

As another application of (20) we shall prove

Theorem 4. *Let M be a complex n-dimensional $(n > 1)$ indefinite l.c.K. manifold with a parallel Lee form $(\nabla\omega = 0)$ and $c = g(B, B) \in \mathbb{R}$. Let $M(c) = M \setminus \mathrm{Sing}(B)$, an open subset of M. Then*

$$P : x \in M(c) \mapsto \mathbb{R}A_x \oplus \mathbb{R}B_x \subset T_x(M)$$

is an integrable distribution, hence P determines a foliation $\mathcal{F}_c$ of $M(c)$ by real surfaces such that i) *either $c \neq 0$ and then each leaf $L \in M/\mathcal{F}_c$ is Riemannian (with the metric $\mathrm{sign}(c)\ i^*g$, $i : L \hookrightarrow M$) and a totally geodesic surface in (M, g), or* ii) *$c = 0$ and then each leaf $L \in M(c)/\mathcal{F}_c$ is either an isotropic surface (when $n \geq 3$) or a totally lightlike surface (when $n = 2$). Assume that $n \geq 3$. Then the second fundamental form of a leaf $L \in M(0)/\mathcal{F}_0$ with respect to any transversal vector bundle $\mathrm{tr}(T(L)) \to L$ vanishes.*

Proof. Note that $c \neq 0$ yields $M(c) = M$. Moreover (by the very definition of the anti-Lee field) $\{A_x, B_x\}$ are linearly dependent if and

only if $x \in \mathrm{Sing}(B)$. Hence the sum $\mathbb{R}A_x + \mathbb{R}B_x$ is direct, for any $x \in M(c)$. To see that P is involutive it suffices to check that $[A, B] \in P$. Let $X \in P^{\perp}$. Then

$$g([A,B],X) = g(\nabla_A B - \nabla_B A, X) = \qquad (\text{as } \nabla B = 0,\ \nabla g = 0)$$
$$= -B(g(A,X)) + g(A, \nabla_B X) = \qquad (\text{as } \theta(X) = 0)$$
$$= -g(JB, \nabla_B X) = g(B, J\nabla_B X) = \qquad (\text{by } (20))$$
$$= g(B, \nabla_B JX) = B(g(B, JX)) - g(\nabla_B B, JX) = B(\theta(X)) = 0,$$

hence $[A, B] \in (P^{\perp})^{\perp} = P$. By the classical Frobenius theorem there is a foliation $\mathcal{F}_c$ of $M(c)$ such that $P = T(\mathcal{F}_c)$. Assume that $c \neq 0$. Then either P is spacelike (when $c > 0$) or timelike (when $c < 0$). Let $L \in M/\mathcal{F}_c$ and let h be the second fundamental form of $L \hookrightarrow M$. Then $\nabla B = 0$ yields $h(A, B) = h(B, B) = 0$. Finally (by (20))

$$\nabla_A A = -\nabla_A JB =$$
$$= -J\nabla_A B - \frac{1}{2}\{\theta(B)A - \omega(B)JA - g(A,B)A - \Omega(A,B)B\} = 0$$

so that $h(A, A) = 0$. We may conclude that $h = 0$. □

Assume now that $c = 0$, so that both the Lee and anti-Lee fields are lightlike. Let us set

$$\mathrm{Rad}\ P_x = P_x \cap P_x^{\perp}, \quad x \in M(0).$$

We have $\dim_{\mathbb{R}} P_x = 2$ and $\dim_{\mathbb{R}} P_x^{\perp} = 2(n-1)$, hence $\mathrm{Rad}\ P = P$. Therefore each leaf $L \in M(0)/\mathcal{F}_0$ is a 2-lightlike submanifold (surface) in $M(0)$ and in particular (according to the terminology in [10], p. 149-150) an isotropic submanifold (when $n > 2$) or a totally lightlike submanifold (when $n = 2$) of $M(0)$. We shall need the following adaptation of Lemma 1.4 and Theorem 1.6 in [10], p. 149-150 (to the case of the lightlike foliation $\mathcal{F}_0$, rather than a single istropic submanifold)

Lemma 6. *Assume that $n \geq 3$. Let $S(P^{\perp}) \to M(0)$ be a vector subbundle of $P^{\perp} \to M(0)$ such that*

$$P^{\perp} = P \oplus_{\mathrm{orth}} S(P^{\perp}). \tag{22}$$

Then for any $x \in M(0)$ there exist an open neighborhood $U \subseteq M(0)$ and a system of linearly independent tangent vector fields $\{N_1, N_2\}$ on U such that

$$\theta(N_1) = \omega(N_2) = 1, \quad \theta(N_2) = \omega(N_1) = 0, \tag{23}$$

$$g(N_i, N_j) = 0, \quad g(N_i, W) = 0, \tag{24}$$

for any $W \in S(P^\perp)$. Moreover, if $\operatorname{ltr}(P)\big|_U$ is given by

$$\operatorname{ltr}(P)_x = \mathbb{R}N_{1,x} \oplus \mathbb{R}N_{2,x}\,, \quad x \in U,$$

then the vector bundles $\operatorname{ltr}(P)\big|_U$ glue up to a vector bundle $\operatorname{ltr}(P) \to M(0)$ such that

$$S(P^\perp)^\perp = P \oplus \operatorname{ltr}(P). \tag{25}$$

According to the terminology in [10], $\operatorname{ltr}(P)\big|_L$ is the *lightlike transversal vector bundle* of $(L, S(P^\perp)\big|_L)$, for each leaf $L \in M(0)/\mathcal{F}_0$. By Proposition 2.1 in [10], p. 5, $S(P^\perp)$ is nondegenerate. Let $S(P^\perp)^\perp$ be the orthogonal complement of $S(P^\perp)$. Of course, this is also nondegenerate and

$$T(M(0)) = S(P^\perp) \oplus_{\mathrm{orth}} S(P^\perp)^\perp. \tag{26}$$

Note that $S(P^\perp)^\perp$ has rank 4 and (by (22)) $P \subset S(P^\perp)^\perp$. To prove Lemma 6 let $E \to M(0)$ be a subbundle of $S(P^\perp)^\perp \to M(0)$ such that

$$S(P^\perp)^\perp = P \oplus E. \tag{27}$$

For any $x \in M(0)$ there is an open neighborhood $U \subseteq M(0)$ of x and a local frame $\{V_1, V_2\} \subset \Gamma^\infty(U, E)$. Let $D \in C^\infty(U)$ be given by

$$D = \theta(V_1)\omega(V_2) - \omega(V_1)\theta(V_2).$$

We claim that $D(x) \neq 0$, for any $x \in U$. Indeed, if $D(x_0) = 0$ for some $x_0 \in U$ then

$$\theta(V_2)_{x_0} = \lambda\theta(V_1)_{x_0}\,, \quad \omega(V_2)_{x_0} = \lambda\omega(V_1)_{x_0}\,,$$

for some $\lambda \in \mathbb{R}$, hence $v_\lambda := V_{2,x_0} - \lambda V_{1,x_0}$ is orthogonal to both the Lee and anti-Lee vectors, i.e. $v_\lambda \in P^\perp_{x_0}$. Yet $v_\lambda \in E_{x_0} \subset S(P^\perp)^\perp_{x_0}$, i.e. v_λ is orthogonal to $S(P^\perp)_{x_0}$. Then (by (22)) $v_\lambda \in P_{x_0} \cap E_{x_0} = (0)$, i.e. $\{V_{1,x_0}, V_{2,x_0}\}$ are linearly dependent, a contradiction. Let us set

$$N_1 = \lambda_{11}A + \lambda_{12}B + \frac{1}{D}\{\omega(V_2)V_1 - \omega(V_1)V_2\},$$

$$N_2 = \lambda_{21}A + \lambda_{22}B - \frac{1}{D}\{\theta(V_2)V_1 - \theta(V_1)V_2\},$$

where $\lambda_{ij} \in C^\infty(U)$ are given by

$$\lambda_{11} = -\frac{1}{D^2}\{\omega(V_2)^2 g(V_1, V_1) - 2\omega(V_1)\omega(V_2)g(V_1, V_2) + \omega(V_1)^2 g(V_2, V_2)\},$$

$$\lambda_{22} = -\frac{1}{D^2}\{\theta(V_2)^2 g(V_1, V_1) - 2\theta(V_1)\theta(V_2)g(V_1, V_2) + \theta(V_1)^2 g(V_2, V_2)\},$$

$$\lambda_{12} = \lambda_{21} = \frac{1}{2D^2}\{\omega(V_2)\theta(V_2)g(V_1, V_1)+$$
$$+[\theta(V_1)\omega(V_2) + \omega(V_1)\theta(V_2)]g(V_1, V_2) - \omega(V_1)\theta(V_1)g(V_2, V_2)\}.$$

A calculation shows that N_i are linearly independent at each $x \in U$ and satisfy (23)-(24). Therefore $\mathrm{ltr}(P)|_U$ is well defined. Let $U' \subseteq M(0)$ be another open neighborhood of x and $\{V_1', V_2'\}$ a local frame of E on U', so that $V_i' = f_i^j V_j$, for some $f_i^j \in C^\infty(U \cap U')$. A calculation shows that $\lambda_{ij}' = \lambda_{ij}$ and then $N_i' = N_i$ on $U \cap U'$, hence $\mathrm{ltr}(P)|_U$ and $\mathrm{ltr}(P)|_{U'}$ glue up over $U \cap U'$. Finally, one may check that $P_{x_0} \cap \mathrm{ltr}(P)_{x_0} \neq (0)$ at some $x_0 \in U$ yields $D(x_0) = 0$, a contradiction. Hence the sum $P + \mathrm{ltr}(P)$ is direct and (25) must hold. $\square$

With the choices in Lemma 6 we set

$$\mathrm{tr}(P) = \mathrm{ltr}(P) \oplus_{\mathrm{orth}} S(P^\perp),$$

so that (cf. [2]) $\mathrm{tr}(P)|_L$ is the *transversal bundle* of $L \in M(0)/\mathcal{F}_0$. Then (26) yields $T(M(0)) = P \oplus \mathrm{tr}(P)$ and we may decompose $\nabla_X Y = \nabla_X^P Y + h^P(X, Y)$, for any $X, Y \in P$, such that ∇^P is a connection in $P \to M(0)$ and h^P is a $C^\infty(M(0))$-bilinear symmetric $\mathrm{tr}(P)$-valued form on P (compare to (2.1) in [2], p. 154). Once again we may use $\nabla B = 0$, $\nabla_A A = 0$ to conclude that $h^P = 0$. $\square$

5. A CR extension result

Let M be a complex n-dimensional indefinite l.c.K. manifold of index $2s$, $0 < s < n$, with a parallel Lee form. Let $\mathcal{F}$ be the first canonical foliation (given by $\omega = 0$). Each leaf L of $\mathcal{F}$ is a real hypersurface in M, hence a CR manifold with the CR structure

$$T_{1,0}(L) = T^{1,0}(M) \cap [T(L) \otimes \mathbb{C}]$$

induced by the complex structure of M. There is a natural first order differential operator

$$\overline{\partial}_L : C^1(L) \to \Gamma^\infty(T_{0,1}(L)^*)$$

given by $(\overline{\partial}_L f)\overline{Z} = \overline{Z}(f)$, for any C^1 function $f : L \to \mathbb{C}$ and any $Z \in T_{1,0}(L)$. Here $T_{0,1}(L) = \overline{T_{1,0}(L)}$. The solutions to $\overline{\partial}_L f = 0$ are the CR functions on L (and $\overline{\partial}_L f = 0$ are the tangential Cauchy-Riemann equations on L, cf. e.g. [4], p. 124). Let $CR^k(L)$ be the space of all CR functions on L, of class C^k. It is a natural question whether a CR function on a leaf L of $\mathcal{F}$ extends to a holomorphic function on M (at least locally). We answer this question for the canonical foliation of Ω_+ (a similar result holds for Ω_-) where an explicit description of the leaves is available. Precisely

Theorem 5. *Let $n \geq 2$ and $0 < s < n$ such that $s \neq (n-1)/2$ when n is odd. Let $w \in S^1$ and $N_w^+ = F^{-1}(S_{2s}^{2n-1} \times \{w\})$, where $F : \Omega_+ \to S_{2s}^{2n-1} \times S^1$ is the diffeomorphism* (11). *Let $\mathcal{F}$ be the foliation of Ω_+ given by the Pfaff equation $d \log |z|_{s,n}^2 = 0$. Then the leaf space is*

$$\Omega_+/\mathcal{F} = \{N_w^+ : w \in S^1\} \tag{28}$$

and for any point $x \in N_w^+$ there is an open neighborhood U of x in M such that for any $f \in CR^1(N_w^+)$ there is a holomorphic function $F \in \mathcal{O}(U)$ such that $F|_{U \cap N_w^+} = f$.

Proof. Note that

$$N_w^+ = \{\pi(\lambda^{\arg(w)/(2\pi)}\zeta) : \zeta \in S_{2s}^{2n-1}\} \quad (w \in S^1).$$

Let $z \in \Lambda_+ \setminus \Lambda_0$ and let us consider $\zeta = |z|_{s,t}^{-1} z \in S_{2s}^{2n-1}$ and $w = \exp(2\pi i \log |z|_{s,n}/\log \lambda) \in S^1$. Then $\arg(w) = 2\pi \log |z|_{s,n}/\log \lambda + 2m\pi$ for some $m \in \mathbb{Z}$, so that

$$\pi(z) = \pi(\lambda^m |z|_{s,t}\zeta) = \pi(\lambda^{\arg(w)/(2\pi)}\zeta) \in N_w^+ ,$$

that is through each point $\pi(z) \in \Omega_+$ passes at least one hypersurface of the form N_w^+. Next, let us assume that $\pi(z) \in N_w^+ \cap N_{w'}^+$. Then

$$e^{\arg(w')/(2\pi)}\zeta' = \lambda^m e^{\arg(w)/(2\pi)}\zeta$$

for some $\zeta, \zeta' \in S_{2s}^{2n-1}$ and some $m \in \mathbb{Z}$. Then $b_{s,n}(\zeta, \zeta) = b_{s,n}(\zeta', \zeta') = 1$ imply

$$\arg(w') = \arg(w) + 2m\pi \log \lambda$$

hence $N_w^+ = N_{w'}^+$, that is through each $\pi(z) \in \Omega_+$ passes a unique hypersurface of the form N_w^+. To emphasize, $N_w^+ = N_{w'}^+$ if and only if $w' = e^{2m\pi i \log \lambda} w$, for some $m \in \mathbb{Z}$. Therefore, to prove (28) it suffices to check that the Lee field B of Ω_+ is orthogonal to each N_w^+. We set

$$D(0, r) = \{z \in \Lambda_+ \setminus \Lambda_0 : |z|_{s,n} < r\} \quad (r > 0)$$

and consider the annulus $A_k = D(0, \lambda^k) \setminus \overline{D}(0, \lambda^{k+1})$, $k \in \mathbb{Z}$. If $U_k = \pi(A_k)$ then $\phi_k = (\pi : A_k \to U_k)^{-1}$ are local charts on Ω_+. Note that the holomorphic transformation F_λ maps the pseudosphere $S_{2s}^{2n-1}(\lambda^k)$ onto $S_{2s}^{2n-1}(\lambda^{k+1})$, for any $k \in \mathbb{Z}$. In other words, when building Ω_+ one identifies the points where a generic complex line through the origin intersects the pseudospheres $S_{2s}^{2n-1}(\lambda^k)$. In particular $\pi(S_{2s}^{2n-1}(\lambda^k)) = \pi(S_{2s}^{2n-1})$ and $U_k = U_0$, for any $k \in \mathbb{Z}$.

Lemma 7. *Let $w \in S^1$ and $a = \arg(w)/(2\pi \log \lambda)$. If $a \in \mathbb{R} \setminus \mathbb{Z}$ then $N_w^+ \subset U_0$, while if $a \in \mathbb{Z}$ then $N_w^+ = \pi(S_{2s}^{2n-1})$. In particular, for any $w \in S^1 \setminus \{e^{2m\pi i \log \lambda} : m \in \mathbb{Z}\}$*

$$\phi_0(N_w^+) = S_{2s}^{2n-1}\left(\lambda^{-[a]} e^{\arg(w)/(2\pi)}\right). \tag{29}$$

Here $[a]$ is the integer part of $a \in \mathbb{R}$. Note that (29) doesn't apply to the leaf $L_0 = \pi(S_{2s}^{2n-1}) \in \Omega_+/\mathcal{F}$ (corresponding to $a \in \mathbb{Z}$). However, in this case one may consider $\lambda < \epsilon < 1$ and the annulus $A = D(0, \lambda^{-1}\epsilon) \setminus \overline{D}(0, \epsilon)$ and then L_0 is contained in $U = \pi(A)$ and $(\pi : A \to U)^{-1}$ is a local chart on Ω_+. To prove Lemma 7 let x be a point of N_w^+, $x = \pi(e^{\arg(w)/(2\pi)}\zeta)$, and let us set $z = \lambda^{k-[a]} e^{\arg(w)/(2\pi)}\zeta$. Then $a - 1 < [a] \le a$ yields $\lambda^{k+1} < |z|_{s,n} \le \lambda^k$ and the second inequality becomes an equality if and only if $a \in \mathbb{Z}$, that is if $w = e^{2m\pi i \log \lambda}$, for some $m \in \mathbb{Z}$. Finally, if $a \in \mathbb{R} \setminus \mathbb{Z}$ then $N_w^+ \subset U_0$ and

$$\phi_0(N_w^+) = \{\phi_0(\pi(^{\arg(w)/(2\pi)}\zeta)) : \zeta \in S_{2s}^{2n-1}\} =$$

$$= \{\lambda^{-[a]} e^{\arg(w)/(2\pi)}\zeta : \zeta \in S_{2s}^{2n-1}\} = S_{2s}^{2n-1}\left(\lambda^{-[a]} e^{\arg(w)/(2\pi)}\right)$$

and $B = -2(z^j Z_j + \overline{z}^j \overline{Z}_j)$ is orthogonal to any $S_{2s}^{2n-1}(r)$. The Cayley transform

$$\mathcal{C}(z) = \left(\frac{z'}{r + z_n}, \frac{i(r - z_n)}{r + z_n}\right), \quad z = (z', z_n) \in \mathbb{C}^n \setminus \{z_n + r = 0\},$$

is a CR isomorphism of $S_{2s}^{2n-1}(r)$ onto $\partial \mathcal{S}_{s,n} \setminus \{\zeta_n + i = 0\}$, where

$$\mathcal{S}_{s,n} = \{\zeta \in \mathbb{C}^n : \mathrm{Im}(\zeta_n) > \sum_{\alpha=1}^{n-1} \epsilon_\alpha |\zeta_\alpha|^2\}.$$

The CR structure $T_{1,0}(\partial \mathcal{S}_{s,n})$ is the span of $\{\partial/\partial \zeta^\alpha + 2i\epsilon_\alpha \overline{\zeta}_\alpha \partial/\partial \zeta^n : 1 \le \alpha \le n-1\}$ hence the Levi form has signature $(s, n-s-1)$. Yet $s \ge 1$ hence (by H. Lewy's CR extension theorem, cf. e.g. Theorem 1 in [4], p. 198) for any $x \in \partial \mathcal{S}_{s,n}$ there is an open neighborhood $U \subseteq \mathbb{C}^n$ of x such that for any $f \in CR^1(\partial \mathcal{S}_{s,n})$ there is a unique $F \in \mathcal{O}(U \cap \mathcal{S}_{s,n}) \cap C^0(U \cap \overline{\mathcal{S}}_{s,n})$ such that $F|_{U \cap \partial \mathcal{S}_{s,n}} = f$. Then the last statement in Theorem 5 holds for any $x \in N_w^+ \setminus \{\pi(e^{\arg(w)/(2\pi)}\zeta) : \zeta \in \Lambda_0^{n-1} \times \{-1\}\}$, where $\Lambda_0^{n-1} = \Lambda^{n-1} \cup \{0\}$ and Λ^{n-1} is the null cone in $\mathbb{C}_s^{n-1}$ (so that $\phi_0(x)$ satisfies $z_n + r \neq 0$ $(r = \lambda^{-[a]} e^{\arg(w)/(2\pi)})$). For arbitrary $x \in N_w^+$ the argument requires that $z_j + r \neq 0$, for some $1 \le j \le n$ (the remaining case is ruled out by our assumption that $n \neq 2s + 1$). $\square$

6. Levi foliations

Let (M, J, g) be a complex n-dimensional indefinite l.c.K. manifold and B, A its Lee and anti-Lee fields, respectively. Let us set $Z := B + iA \in T^{1,0}(M)$. Clearly $\omega(Z) = c$. Let us assume that $\nabla\omega = 0$ and $\mathrm{Sing}(B) = \emptyset$ and set

$$T_{1,0}(\mathcal{F}) = T^{1,0}(M) \cap [T(\mathcal{F}) \otimes \mathbb{C}]$$

so that the portion of $T_{1,0}(\mathcal{F})$ over a leaf $L \in M/\mathcal{F}$ is the CR structure of L. Also the portion of $H(\mathcal{F}) := \mathrm{Re}\{T_{1,0}(\mathcal{F}) \oplus \overline{T_{1,0}(\mathcal{F})}\}$ over L is the Levi distribution $H(L)$ of L. The distribution $H(\mathcal{F})$ carries the complex structure

$$J : H(\mathcal{F}) \to H(\mathcal{F}), \quad J(V + \overline{V}) = i(V - \overline{V}), \quad V \in T_{1,0}(\mathcal{F}).$$

See also [9]. Let us set

$$\mathcal{L}(V, \overline{W}) = i\, \pi\, [V, \overline{W}], \quad V, W \in T_{1,0}(\mathcal{F}),$$

where $\pi : T(\mathcal{F}) \to T(\mathcal{F})/H(\mathcal{F})$ is the natural projection, so that $\mathcal{L}$ is the Levi form of each leaf of $\mathcal{F}$. The *null space* of $\mathcal{L}$ is

$$\mathrm{Null}(\mathcal{L}) = \{V \in T_{1,0}(\mathcal{F}) : \mathcal{L}(V, \overline{V}) = 0\}.$$

We may state the following corollary of Theorems 1 and 4

Proposition 4. *If the Lee vector B is lightlike then the Levi form of each leaf of $\mathcal{F}$ is degenerate ($Z \in \mathrm{Null}(\mathcal{L})$) and $\mathcal{F}_0$ is a subfoliation of $\mathcal{F}$. Moreover if $n = 2$ then each leaf of $\mathcal{F}$ is Levi-flat and the Levi foliation of each leaf $L \in M/\mathcal{F}$ extends to a unique holomorphic foliation of M.*

We recall that a CR manifold L is *Levi-flat* if its Levi form vanishes identically ($\mathcal{L} = 0$). If this is the case L is foliated by complex manifolds (whose complex dimension equals the CR dimension of L). The resulting foliation (whose tangent bundle if the Levi distribution $H(L)$ of L) is the *Levi foliation* of L. If L is embedded in some come complex manifold M a problem raised by C. Rea, [14], is whether the Levi foliation of (a Levi-flat CR manifold) L may extend to a holomorphic foliation of M. Proposition 4 exhibits a family of Levi foliations of class C^∞ which extend holomorphically (while Rea's extension theorem (cf. *op. cit.*) requires real analytic data).

Proof of Proposition 4. Let us assume that $c = 0$. Then $Z \in T(\mathcal{F}) \otimes \mathbb{C}$ hence $Z \in T_{1,0}(\mathcal{F})$. Moreover $Z + \overline{Z} \in H(\mathcal{F})$ yields $B \in H(\mathcal{F})$ and by applying J we may conclude that $A \in H(\mathcal{F})$ as well. Hence (with the notations of Theorem 4) $P \subseteq H(\mathcal{F})$. Note that $[Z, \overline{Z}] = 2i[A, B]$ and then (by the integrability of P) $\mathcal{L}(Z, \overline{Z}) = 0$. When $n = 2$ each

leaf L of $\mathcal{F}$ is a 3-dimensional CR manifold hence $H(\mathcal{F}) = P$ and L is Levi-flat. Finally the foliation induced by $\mathcal{F}_0$ on L is precisely the Levi foliation of L. In other words, the Levi foliation of each leaf of $\mathcal{F}$ extends to a holomorphic foliation of M which is precisely the second canonical foliation of M.

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